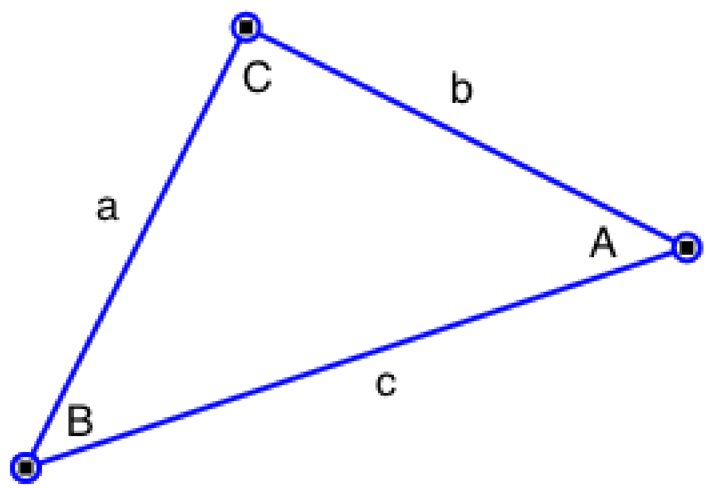
MATH 32 FALL 2012
FINAL EXAM - PRACTICE EXAM SOLUTIONS

(1) (6 points) Solve the equation $|x-1| = 3$.

Solution: Since $|x-1| = 3$, $x - 1 = 3$ or $x - 1 = -3$. Solving for x, $x = 4$ or $x = -2$.

(2) In the triangle below, let $a = 4$, $b = 2$, and $c = \sqrt{22}$.



(a) (3 points) Find $\cos(C)$.

Solution: The law of cosines says $a^2 + b^2 - 2ab\cos(C) = c^2$. So

$$\begin{aligned} 4^2 + 2^2 - 2(4)(2)\cos(C) &= \sqrt{22}^2 \\ 20 - 16\cos(C) &= 22 \\ -16\cos(C) &= 2 \\ \cos(C) &= -\frac{2}{16} = -\frac{1}{8}. \end{aligned}$$

(b) (3 points) Find the area of the triangle.

Solution: The area of the triangle is $\frac{1}{2}ab\sin(C)$, so we need to find $\sin(C)$.

$$\sin^2(C) + \cos^2(C) = 1$$
$$\sin^2(C) + (-\frac{1}{8})^2 = 1$$
$$\sin^2(C) = 1 - \frac{1}{64}$$
$$\sin(C) = \pm\sqrt{\frac{63}{64}} = \frac{\sqrt{63}}{8}.$$

Here we take the positive value because $0 < \theta < \pi$, and hence $\sin(\theta) > 0$.
So the area is $\frac{1}{2}(2)(4)\frac{\sqrt{63}}{8} = \frac{\sqrt{63}}{2}$.

(3) (6 points) Sketch a graph of $y = x^4 + 3x^3 + 2x^2$. Label the x-intercepts.

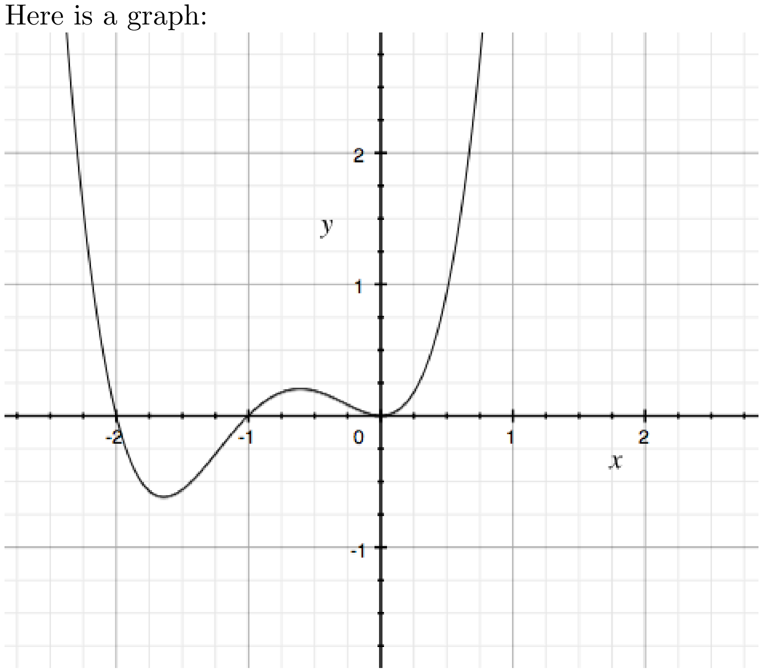
Solution: Factoring, we have $y = x^2(x+1)(x+2)$. So the zeros of the polynomial are 0, -1, and -2, and the x-intercepts are $(-2, 0)$, $(-1, 0)$, and $(0, 0)$.

Next we'll do some sign analysis.

	$(-\infty, -2)$	$(-2, -1)$	$(-1, 0)$	$(0, \infty)$
x^2	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$
Total:	$+$	$-$	$+$	$+$

Finally, since this is a polynomial which is positive as x gets large and negative and as x gets large and positive, the values of the function to go $+\infty$ in both directions.

Here is a graph:



(4) (6 points) Solve the equation $2^x = 8^{2x-4}$.

Solution: We'll take the logarithm base 2 of both sides:

$$\begin{aligned}
\log_2 2^x &= \log_2 8^{2x-4} \\
x &= (2x-4)\log_2 8 \\
x &= (2x-4)\cdot 3 \\
x &= 6x-12 \\
-5x &= -12 \\
x &= \frac{12}{5}
\end{aligned}$$

(5) (6 points) Find two points (x, y) on the line $y = 2x + 3$ which are at distance 3 from the origin $(0, 0)$.

Solution: A point on the line $y = 2x + 3$ has the form $(x, 2x + 3)$. Using the distance formula, the following equation expresses that this point is at distance 3 from the origin:

$$\begin{aligned}
\sqrt{(x-0)^2 + ((2x+3)-0)^2} &= 3 \\
x^2 + (2x+3)^2 &= 9 \\
x^2 + 4x^2 + 12x + 9 &= 9 \\
5x^2 + 12x &= 0 \\
5x(x + \frac{12}{5}) &= 0
\end{aligned}$$

So $x = 0$ or $x = -\frac{12}{5}$.
If $x = 0$, $y = 2(0) + 3 = 3$. If $x = -\frac{12}{5}$, $y = 2\frac{-12}{5} + 3 = -\frac{24}{5} + \frac{15}{5} = -\frac{9}{5}$.
So the two points are $(0, 3)$ and $(-\frac{12}{5}, -\frac{9}{5})$.

(6) (6 points) Suppose that $\sin(\theta) = -\frac{2}{3}$ and $\frac{3\pi}{2} < \theta < 2\pi$. Find $\cos(\frac{\theta}{2})$.

Solution: We need to use the half-angle formula for cosine, $\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$. So first we need to find $\cos(\theta)$.

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
(-\frac{2}{3})^2 + \cos^2(\theta) &= 1 \\
\cos^2(\theta) &= 1 - \frac{4}{9} \\
\cos(\theta) &= \pm\sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}
\end{aligned}$$

Here we take the positive value, because $\frac{3\pi}{2} < \theta < 2\pi$, i.e. θ is in the 4th quadrant, so $\cos(\theta) > 0$. Now:

$$\begin{aligned}\cos(\frac{\theta}{2}) &= \pm\sqrt{\frac{1+\cos(\theta)}{2}} \\ &= \pm\sqrt{\frac{1+\frac{\sqrt{5}}{3}}{2}} \\ &= \pm\sqrt{\frac{\frac{3+\sqrt{5}}{3}}{2}} \\ &= \pm\sqrt{\frac{3+\sqrt{5}}{6}}\end{aligned}$$

Now we need to choose a sign. Since $\frac{3\pi}{2} < \theta < 2\pi$, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, so $\frac{\theta}{2}$ is in the 2nd quadrant, and $\cos(\frac{\theta}{2})$ is negative. So

$$\cos(\frac{\theta}{2}) = -\sqrt{\frac{3+\sqrt{5}}{6}}.$$

(7) (6 points) Suppose a bank promises that money placed in a certain account will triple in 20 years. Assuming that interest is compounded continuously, what interest rate must the bank offer to make good on their promise?

Solution: Interest is compounded continuously, so we use the formula $A = Pe^{rt}$. Here $t = 20$ (the time in years), $A = 3P$ (we're tripling the principal), and r is the quantity we'd like to find. So

$$\begin{aligned}3P &= Pe^{r\cdot 20} \\ 3 &= e^{20r} \\ \ln(3) &= \ln(e^{20r}) \\ \ln(3) &= 20r \\ r &= \frac{\ln(3)}{20}\end{aligned}$$

(8) (6 points) Show that for all θ, $\tan(\theta)\sin(\theta) = \sec(\theta) - \cos(\theta)$.

Solution: I'll start with the left hand side.

$$\begin{aligned}\tan(\theta)\sin(\theta) &= \frac{\sin(\theta)}{\cos(\theta)}\sin(\theta)\\ &= \frac{\sin^2(\theta)}{\cos(\theta)}\\ &= \frac{1-\cos^2(\theta)}{\cos(\theta)}\\ &= \frac{1}{\cos(\theta)} - \frac{\cos^2(\theta)}{\cos(\theta)}\\ &= \sec(\theta) - \cos(\theta)\end{aligned}$$

(9) (6 points) Find all values of x satisfying

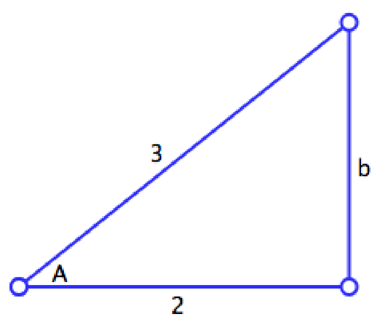
$$\sin^{-1}(x) = \cos^{-1}\left(\frac{2}{3}\right) + \frac{\pi}{3}.$$

Express your answer without using any trig or inverse trig functions.

Solution: Take sin of both sides. We'll use the fact that $\sin(\sin^{-1}(x)) = x$, $\cos(\cos^{-1}(x)) = x$, and the angle sum formula for sin.

$$\begin{aligned}\sin(\sin^{-1}(x)) &= \sin(\cos^{-1}(\frac{2}{3}) + \frac{\pi}{3})\\ x &= \sin(\cos^{-1}(\frac{2}{3}))\cos(\frac{\pi}{3}) + \cos(\cos^{-1}(\frac{2}{3}))\sin(\frac{\pi}{3})\\ &= \sin(\cos^{-1}(\frac{2}{3}))(\frac{1}{2}) + (\frac{2}{3})(\frac{\sqrt{3}}{2}).\end{aligned}$$

It remains to find $\sin(\cos^{-1}(\frac{2}{3}))$. To do this, we draw a triangle:



Here $\cos(A) = \frac{2}{3}$, so $A = \cos^{-1}(\frac{2}{3})$. We have $2^2 + b^2 = 3^2$, so $b^2 = 9 - 4$, and $b = \sqrt{5}$. Then $\sin(\cos^{-1}(\frac{2}{3})) = \sin(A) = \frac{\sqrt{5}}{3}$. So:

$$\begin{aligned}x &= (\frac{\sqrt{5}}{3})(\frac{1}{2}) + (\frac{2}{3})(\frac{\sqrt{3}}{2})\\ &= \frac{\sqrt{5}}{6} + \frac{2\sqrt{3}}{6}\\ &= \frac{\sqrt{5}+2\sqrt{3}}{6}.\end{aligned}$$

(10) (6 points) Find the vertex of the parabola given by $y = 2x^2 + 8x + 9$.

Solution: Completing the square,

$$\begin{aligned} y &= 2(x^2 + 4x) + 9 \\ &= 2((x+2)^2 - 4) + 9 \\ &= 2(x+2)^2 - 8 + 9 \\ &= 2(x+2)^2 + 1. \end{aligned}$$

So the vertex is $(-2, 1)$.

(11) Let $f(x) = 3\ln(x) + \ln(\frac{1}{x})$.

(a) (3 points) What are the domain and range of f? To find the range, it may be helpful to simplify the formula for f a bit.

Solution: Domain: $(0, \infty)$.
This is because the domain of $\ln(x)$ is $(0, \infty)$, and if $x > 0$, $\frac{1}{x} > 0$, so there are no problems as long as x is positive.
To see the range, we'll rewrite $f(x)$.

$$\begin{aligned} f(x) &= 3\ln(x) + \ln(\frac{1}{x}) \\ &= \ln(x^3) + \ln(\frac{1}{x}) \\ &= \ln(x^3 \cdot \frac{1}{x}) \\ &= \ln(x^2) \\ &= 2\ln(x). \end{aligned}$$

Since the range of $\ln(x)$ is $(-\infty, \infty)$, doubling all the output values still gives $(-\infty, \infty)$. You can also see this by viewing the graph of $f(x)$ as a vertical stretch by 2 of the graph of $\ln(x)$.
Range: $(-\infty, \infty)$.

(b) (3 points) Find a formula for f^{-1}.

Solution: Set $x = 3\ln(y) + \ln(\frac{1}{y})$ and solve for y.
We'll rewrite the right-hand side exactly as above to get

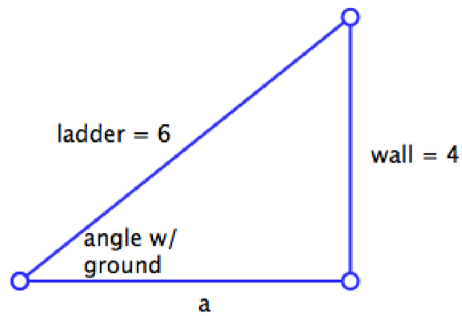
$$\begin{aligned} x &= 2\ln(y) \\ \frac{x}{2} &= \ln(y) \\ y &= e^{\frac{x}{2}} = (e^x)^{\frac{1}{2}} = \sqrt{e^x}. \end{aligned}$$

So $f^{-1}(x) = \sqrt{e^x}$ or $f^{-1}(x) = e^{\frac{x}{2}}$, whichever you prefer.

(12) You rest a 6-foot-tall ladder against a 4-foot-tall fence so that the top of the ladder meets the top of the fence.

(a) (3 points) What angle does the ladder make with the ground?

Solution: Here is a picture:



Calling the angle with the ground θ, $\sin(\theta) = \frac{4}{6}$, so $\theta = \sin^{-1}(\frac{4}{6}) = \sin^{-1}(\frac{2}{3})$.

(b) (3 points) How far away from the base of the wall is the base of the ladder?

Solution: We need to find the side lenght labeled a in the picture. By the Pythagorean theorem, $a^2 + 4^2 = 6^2$, so $a^2 = 36 - 16$, and $a = \sqrt{20}$ or $2\sqrt{5}$ ft.

(13) (6 points) What is 3° in radians?

Solution: $3° = 3° \cdot \frac{\pi}{180°} = \frac{3\pi}{180} = \frac{\pi}{60}$ radians.

(14) (6 points) Find the domain of the following function:

$$f(x) = \frac{\sqrt{2+x}}{\ln(1-x)}.$$

Solution:

- We can't take the square root of a negative number. We must have $2 + x \geq 0$, so $x \geq -2$.
- We can only take the natural log of a positive number. We must have $1 - x > 0$, so $-x > -1$, and $x < 1$.
- We can't divide by 0. The denominator is 0 when $\ln(1 - x) = 0$, which is when $1 - x = e^0 = 1$, when $x = 0$. So 0 must be excluded from the domain.

Putting this together, we have as our domain $[-2, 0) \cup (0, 1)$.

(15) (6 points) Give an example of a periodic function with amplitude $\frac{1}{2}$ and period 3 whose graph goes through the point $(0, 2)$.

Solution: There are many ways to answer this, but here's an example. I'll start with sin as a base function, because I know it is periodic with period 2π and amplitude 1, and it goes through $(0, 0)$.

I'll write down a function of the form $f(x) = a\sin(hx) + v$.
The amplitude of a function of this form is a, so $a = \frac{1}{2}$.
The period is $\frac{2\pi}{h}$, so $\frac{2\pi}{h} = 3$, $2\pi = 3h$, $h = \frac{2\pi}{3}$.
And $f(0) = a\sin(h \cdot 0) + v = v$, so $v = 2$.
Here's our function: $f(x) = \frac{1}{2}\sin(\frac{2\pi}{3}x) + 2$.